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Homogeneous radiation-filled universe in general scalar tensor theory

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Abstract. The cosmological equations for the general scalar tensor theory proposed by Nordvedt (1970) in a Bianchi type-I radiation-filled universe are solved and the behaviour of the model is discussed. There are two distinct situations. Either the universe will explode from a big bang type singularity and continuously increase, or the universe may continuously contract to approach the singularity at the end.

1. Introduction

In view of some recent evidence (Van Flandern 1975) in favour of varying gravitational 'constant' G, the interest in studying cosmologies accounting for such a phenomenon has increased. Theories of Dirac (1973), Hoyle and Narlikar (1971), Brans and Dicke (1961) and more recently the scale covariant theory of Canuto *et al* (1977) have been proposed, in which the Newtonian gravitation parameter G varies. A general scalar tensor theory of gravitation, which may be said to be a generalisation of Brans–Dicke theory, has also been discussed in this context by many authors (Bergman 1968, Wagoner 1970, Nordvedt 1970). Bishop (1976) used the formulation of Nordvedt for a homogeneous, isotropic and pressure-free cosmological model where the coupling parameter ω is no longer a constant quantity as in Brans–Dicke theory, but is a function of the scalar field. In the present paper we have used the same theory of gravitation for a radiation-filled Bianchi type-I homogeneous cosmological model (Ryan and Shepley 1975). The equation of state used here is $\rho = 3p$. This equation of state is of particular interest for the description of the dynamics of a 'hot' universe at the radiation and lepton eras. Moreover, there might be a large amount of anisotropy in the early universe in spite of the fact that the present universe appears to be highly isotropic.

We have chosen a relation of the form $\rho \sim \phi^n$, where n may be any number, and finally obtained cosmological models with somewhat different properties at various ranges of the value of n. As in Bishop's work (1976) all of the models discussed here are of the 'big bang' type.

There are two distinct situations. Either the universe will explode from a big bang type singularity and continuously expand or the universe may continuously contract to approach the singularity at the end. There is no bounce anywhere in the history of the models. Somewhat unusual behaviour is observed in one of the expanding models,

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where there is acceleration instead of deceleration in the motion during the evolution of the universe. This is perhaps due to the repulsive effect of the scalar field, which causes the acceleration. A similar but not exactly identical situation is described by Ruban and Finkelstein (1975) for the anisotropic model with matter having an equation of state $p \leq \rho/3$ in Brans–Dicke theory, where a bounce takes place and prevents the singularity development if the parameter ω is negative ($\omega < -6$). They explain the phenomenon as due to the repulsion of the scalar field owing to the negative effective energy density. Such forces opposing the gravitational attractions are also encountered in the presence of a cosmological constant.

2. General scalar tensor theory

The field equations of the general scalar tensor theory (Nordvedt 1970) can be expressed as

$$G_{\mu\nu} = -(k/\phi)T_{\mu\nu} - (\omega/\phi^2)[\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}] - (1/\phi)[\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi] \tag{2.1}$$

$$\Box\phi = (3+2\omega)^{-1}[kT - \phi_{,\lambda}\phi^{,\lambda}(\mathrm{d}\omega/\mathrm{d}\phi)]. \tag{2.2}$$

Where $k = 8\pi G_0$, and G_0 is an arbitrary constant having no effect on the physical results of the theory, so we can choose $G_0 = G$ (today) and ϕ becomes a dimensionless scalar field. It can be shown that as a consequence of (2.1) and (2.2) we have for matter energy momentum tensor the following relation,

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{2.3}$$

The gravitational constant in this theory measured by the observation of a slowly moving particle or in time dilation experiments is

$$G = G_0\left(\frac{2\omega+4}{2\omega+3}\right)\phi^{-1}. \tag{2.4}$$

This can be shown easily by analogy with the post-Newtonian approximation for Einstein's field equations (Weinberg 1972).

We consider the energy momentum tensor for the matter as a perfect fluid, so that

$$T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}, \tag{2.5}$$

where ρ is the mass density, p the pressure and u_μ the four-velocity satisfying

$$u^\mu u_\mu = -1. \tag{2.6}$$

We assume an equation of state in the form

$$p = \tfrac{1}{3}\rho, \tag{2.7}$$

which is appropriate for a radiation-filled universe. For the space–time we choose a spatially homogeneous universe Bianchi type-I having the line element given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{2\gamma(t)}\,\mathrm{d}x^2 + \mathrm{e}^{2\theta(t)}\,\mathrm{d}y^2 + \mathrm{e}^{2\psi(t)}\,\mathrm{d}z^2. \tag{2.8}$$

Using comoving coordinates so that $u^\alpha = \delta^\alpha_0$ the non-vanishing components of the field equations (2.1) with (2.5) and (2.8) are

$$G_0^0 = \tfrac{9}{2}(\dot{R}/R)^2 - \tfrac{1}{2}\omega(\dot{\gamma}^2 + \dot{\theta}^2 + \dot{\psi}^2)$$
$$= (k/\phi)\rho + \tfrac{1}{2}\omega(\dot{\phi}/\phi)^2 + (\ddot{\phi}/\phi) + (\Box\phi/\phi), \tag{2.9}$$
$$G_1^1 = \ddot{\theta} + \ddot{\psi} + \tfrac{3}{2}(\dot{R}/R)[-\dot{\gamma} + \dot{\theta} + \dot{\psi}] + \tfrac{1}{2}[\dot{\gamma}^2 + \dot{\theta}^2 + \dot{\psi}^2]$$
$$= -(k/\phi)p - \tfrac{1}{2}\omega(\dot{\phi}/\phi)^2 + \dot{\gamma}(\dot{\phi}/\phi) + (\Box\phi/\phi), \tag{2.10}$$
$$G_2^2 = \ddot{\gamma} + \ddot{\psi} + \tfrac{3}{2}(\dot{R}/R)[\dot{\gamma} - \dot{\theta} + \dot{\psi}] + \tfrac{1}{2}[\dot{\gamma}^2 + \dot{\theta}^2 + \dot{\psi}^2]$$
$$= -(k/\phi)p - \tfrac{1}{2}\omega(\dot{\phi}/\phi)^2 + \dot{\theta}(\dot{\phi}/\phi) + (\Box\phi/\phi), \tag{2.11}$$
$$G_3^3 = \ddot{\gamma} + \ddot{\theta} + \tfrac{3}{2}(\dot{R}/R)[\dot{\gamma} + \dot{\theta} - \dot{\psi}] + \tfrac{1}{2}[\dot{\gamma}^2 + \dot{\theta}^2 + \dot{\psi}^2]$$
$$= -(k/\phi)p - \tfrac{1}{2}\omega(\dot{\phi}/\phi)^2 + \dot{\psi}(\dot{\phi}/\phi) + (\Box\phi/\phi) \tag{2.12}$$

and the wave equation (2.2) becomes,

$$\Box\phi = -\ddot{\phi} - 3(\dot{R}/R)\dot{\phi} = (3 + 2\omega)^{-1}\dot{\phi}^2(\mathrm{d}\omega/\mathrm{d}\phi). \tag{2.13}$$

The dot means differentiation with respect to time and

$$R^3 = \exp[\gamma + \theta + \psi].$$

Equation (2.13) can be integrated, giving

$$\dot{\phi} = \frac{A}{(2\omega + 3)^{1/2}} \frac{1}{R^3}, \tag{2.14}$$

where A is a constant of integration.

The trace of (2.1) with (2.8) is

$$3(\dot{R}/R)^2 + 6\ddot{R}/R + \dot{\gamma}^2 + \dot{\theta}^2 + \dot{\psi}^2 = -\omega(\dot{\phi}/\phi)^2 + 3(\Box\phi/\phi), \tag{2.15}$$

and equating (2.15) with (2.9) we have

$$12(\dot{R}/R)^2 + 6(\ddot{R}/R) - 2kp/\phi - 2(\ddot{\phi}/\phi) - 5\Box\phi/\phi = 0. \tag{2.16}$$

Equations (2.5) and (2.7) allow us to integrate (2.3)

$$\rho = B/R^4, \tag{2.17}$$

where B is a constant of integration.

Subtracting equation (2.11) from (2.12) we obtain

$$G_3^3 - G_2^2 = (\dot{\psi} - \dot{\theta})(\dot{\phi}/\phi) = \ddot{\theta} - \ddot{\psi} + \dot{\theta}^2 - \dot{\psi}^2 + \dot{\gamma}(\dot{\theta} - \dot{\psi}), \tag{2.18}$$

and assuming $\dot{\psi} - \dot{\theta} \neq 0$ we integrate (2.18) and then we have

$$1/\phi = (\dot{\theta} - \dot{\psi})D_1R^3. \tag{2.19}$$

By an analogous procedure with equations (2.10)–(2.12) and assuming $(\dot{\gamma} - \dot{\psi}) \neq 0$ and $(\dot{\gamma} - \dot{\theta}) \neq 0$ we obtain

$$1/\phi = (\dot{\gamma} - \dot{\psi})D_2R^3 = (\dot{\gamma} - \dot{\theta})D_3R^3, \tag{2.20}$$

where D_1, D_2 and D_3 are constants of integration and satisfy

$$D_1D_2 + D_2D_3 = D_1D_3. \tag{2.21}$$

3. Solutions of the field equations for a particular model

The independent equations that we have to solve are five, (2.9)–(2.12) and (2.13), and the number of unknowns are six, which are γ, θ, ψ, ω, ϕ and ρ, hence we have here the freedom to assume one appropriate relation between these variables to obtain solutions of the system. We make the assumption that the mass density is proportional to an arbitrary power of the scalar field,

$$\rho = b\phi^n \tag{3.1}$$

where b is a positive constant.

The wave equation (2.13) with (2.17) and (3.1) gives

$$-\ddot{\phi}/\dot{\phi} + \tfrac{3}{4}n(\dot{\phi}/\phi) = (2\omega+3)^{-1}(\mathrm{d}\omega/\mathrm{d}t). \tag{3.2}$$

Substituting (3.2) back into the wave equation (2.13) and with the help of (2.17), (2.19), (2.20) and (3.1) we have

$$3\dot{\psi} = -(1/\phi)(b\phi^n/B)^{3/4}(1/D_1 + 1/D_2) - \tfrac{3}{4}n(\dot{\phi}/\phi). \tag{3.3}$$

Equations (2.9) with (2.17), (2.19), (2.20) and (3.1) give

$$\begin{aligned}3\dot{\psi} = &-(1/\phi)(b\phi^n/B)^{3/4}(1/D_1+1/D_2) \pm \{(1/\phi^2)(b\phi^n/B)^{3/2}(1/D_1+1/D_2)^2 \\ &-6(\dot{\phi}/\phi)^2[-\tfrac{1}{2}(\tfrac{3}{4}n)^2 - \tfrac{3}{4} + \tfrac{3}{4}n] - (3/\phi^2)(b\phi^n/B)^{3/2}(1/D_1^2 + 1/D_2^2) \\ &-6kb\phi^{(n-1)} - \tfrac{3}{2}(A/\phi)^2(b\phi^n/B)^{3/2}\}^{1/2}.\end{aligned} \tag{3.4}$$

Equations (3.3) and (3.4) give

$$\tfrac{3}{4}(\tfrac{1}{2}n-1)^2(\dot{\phi}/\phi)^2 + P\phi^{(\frac{3}{2}n-2)} - kb\phi^{(n-1)} = 0, \tag{3.5}$$

where P is a constant given by

$$P = \tfrac{1}{6}(b/B)^{3/2}[(1/D_1+1/D_2)^2 - 3(1/D_1^2+1/D_2^2) - \tfrac{3}{4}A^2]. \tag{3.6}$$

Equations (2.14) together with (2.17), (3.1) and (3.5) give the value of ω in terms of ϕ, which substituted in (2.4) gives

$$G = G_0\left\{\left[1 - \left(\frac{B}{b}\right)^{3/2}\frac{P}{A^2L}\right]\frac{1}{\phi} + \left(\frac{B}{b}\right)^{3/2}\frac{kb}{A^2L}\phi^{-n/2}\right\} \tag{3.7}$$

where

$$L = \tfrac{3}{4}(\tfrac{1}{2}n-1)^2.$$

Now if one solves for ϕ in equation (3.5), one can immediately integrate (3.3) to get the solution for ψ. From (2.17) and (3.1) it is easy to obtain R in terms of ϕ and then integrating (2.19) one can obtain the solution for θ because ϕ and ψ are already known. Similarly γ can be solved from (2.20).

4. Behaviour of the solutions

We can write the integration of (3.5) in terms of elementary functions only for particular values of n (Gradshteyn and Ryzhik 1965). We will not here give the explicit solution for particular values of n, but rather try to study the general behaviour of the solutions.

From (2.17) and (3.1) we have

$$R^4 = (B/b)\phi^{-n}, \tag{4.1}$$

and also rewriting equation (3.5), we have

$$\dot{\phi} = \frac{2}{\sqrt{3}(\frac{1}{2}n - 1)}[kb\phi^{(n+1)} - P\phi^{3n/2}]^{1/2}. \tag{4.2}$$

It is not difficult to show that the constant P given by (3.6) is a negative definite quantity and in consequence from (4.2) we have neither a maximum nor a minimum for ϕ. This means that we have either an expanding universe starting from a big bang type singularity and increasing in size indefinitely without any bounce when $0 < n < 2$, or we have a universe collapsing steadily towards the singularity when $n > 2$ or $n < 0$. In the first case $\dot{\phi} < 0$ always and in the second case $\dot{\phi} > 0$ for $n > 2$, while $\dot{\phi} < 0$ for $n < 0$. These can be easily verified from the relation (4.2). When $n = 2$, however, the solution is not consistent.

It is interesting to note what would happen to the value of G and also the parameter ω near the singularity. One can write, using (2.14), (3.5) and (4.1), the following expression for ω:

$$(2\omega + 3) = \frac{\frac{3}{4}(b/B)^{3/2}A^2(\frac{1}{2}n - 1)^2}{kb\phi^{(1-n/2)} + h^2}. \tag{4.3}$$

In the above we write h^2 for $-P$.

One can now make an estimate for the parameters ω and G near the singularity in different cases.

When $n < 0$ we have from (3.7) and (4.3) near the singularity

$$\phi \to 0, \qquad G \to \infty \qquad \text{and} \quad \omega \to \{[3(b/B)^{3/2}A^2/8h^2](\tfrac{1}{2}n - 1)^2 - \tfrac{3}{2}\}.$$

When $0 < n < 2$ we have similarly near the singularity

$$\phi \to \infty, \qquad G \to 0 \qquad \text{and} \quad \omega \to -\tfrac{3}{2}.$$

When $n > 2$ we have near the singularity

$$\phi \to \infty, \qquad G \to 0 \qquad \text{and} \quad \omega \to \{[3(b/B)^{3/2}A^2/8h^2](\tfrac{1}{2}n - 1)^2 - \tfrac{3}{2}\}.$$

It is clear from above that at least for a certain range of values of n, the parameter ω assumes negative magnitudes near the singularity. However, at later times ω may increase and can attain positive values.

We now compute two other important quantities representing the expansion scalar and the anisotropy of the model. It is interesting to investigate their magnitudes near the singularity, because this gives some understanding about the dynamics of the models for different ranges of n. The expansion scalar $\bar{\theta}$ and the anisotropy σ are defined respectively (Raychaudhuri 1955) as

$$\bar{\theta} = 3\dot{R}/R \tag{4.4}$$

and

$$\sigma^2 = \frac{1}{12}\left[\left(\frac{\dot{g}_{11}}{g_{11}} - \frac{\dot{g}_{22}}{g_{22}}\right)^2 + \left(\frac{\dot{g}_{22}}{g_{22}} - \frac{\dot{g}_{33}}{g_{33}}\right)^2 + \left(\frac{\dot{g}_{33}}{g_{33}} - \frac{\dot{g}_{11}}{g_{11}}\right)^2\right]. \tag{4.5}$$

Using (4.1), (4.2) and also (2.19), (2.20) one can calculate the above scalars to obtain finally

$$\bar{\theta} = -\frac{\sqrt{3}}{2}\frac{n}{(\frac{1}{2}n-1)}[kb\phi^{(n-1)} + h^2\phi^{2(\frac{3}{4}n-1)}]^{1/2}, \tag{4.6}$$

and

$$|\sigma| = \frac{1}{\sqrt{3}}\left(\frac{b}{B}\right)^{3/4}\left[\frac{1}{D_1^2}+\frac{1}{D_2^2}+\frac{1}{D_3^2}\right]^{1/2}\phi^{(\frac{3}{4}n-1)}. \tag{4.7}$$

We have here different situations depending on the value of n. In all cases the singularity occurs when $R \to 0$ and consequently from (2.17) $\rho \to \infty$, which means that the proper volume tends to vanish and the radiation density ρ and the pressure p both approach infinitely large magnitudes.

Case (i): For $n > 2$ we have a collapsing system and at the singularity

$$\phi \to \infty, \qquad \bar{\theta} \to -\infty \qquad \text{and} \qquad |\sigma| \to \infty.$$

Case (ii): For $n < 0$ we have again a collapsing system and at the singularity

$$\phi \to 0, \qquad \bar{\theta} \to -\infty \qquad \text{and} \qquad |\sigma| \to \infty.$$

In all other cases we have an explosion from the singularity where $\phi \to \infty$. These cases are interesting and deserve a more detailed analysis.

Case (iii): $\frac{4}{3} < n < 2$. We have $\bar{\theta} \to \infty$ when $\phi \to \infty$, but at this point $|\sigma| \to \infty$ when $\frac{4}{3} < n < 2$ and $|\sigma|$ approaches a constant finite quantity for $n = \frac{4}{3}$. From (4.2) and (3.3) it is evident that in this case $\dot{\phi} < 0$ and behaves as $\phi^{(n+1)/2}$ when $\phi \to \infty$. Also $\dot{\psi} \sim \phi^{(n-1)/2} \sim 1/t$ by a suitable choice of the integration constant so that at $t = 0$ we have the singularity and $\dot{\psi} \to \infty$. The time behaviour of ψ near $t = 0$ is then easy to calculate. It can be further shown from the simple analysis of equations (2.19) and (2.20) that $\dot{\gamma}$ and $\dot{\theta}$ have the same properties near the singularity, which is of point-like character.

By analogous calculations for small values of ϕ one can show that both the expansion scalar $\bar{\theta}$ and the anisotropy $|\sigma|$ vanish asymptotically at late stages of the evolution.

Case (iv): $1 < n < \frac{4}{3}$. We have $\bar{\theta} \to \infty$ and $|\sigma| \to 0$ when $\phi \to \infty$.

In this case $\dot{\phi} < 0$ and near the singularity $\dot{\phi}$ behaves as $\phi^{(n+1)/2}$, so that $\dot{\psi} \sim \phi^{(n-1)/2} \sim 1/t$. At the singularity $t = 0$ $\dot{\psi}$, $\dot{\theta}$ and $\dot{\gamma}$ all attain infinitely large positive values. Time behaviours of ψ, θ and γ near the singularity can also be determined and are found to represent a point singularity as in case (iii). The asymptotic behaviour is, however, different in this case. Asymptotically at later time $\phi \to 0$ and the expansion scalar $\bar{\theta}$ attains an infinitely large magnitude. The expansion $\bar{\theta}$ is infinitely large at the initial instant of point singularity, decreases in the course of time and again increases at a later period. This is perhaps due to the predominance of the repulsion effect of the scalar field at later stages of the evolution. It is interesting to note that in this case near the big bang singularity the expansion is isotropic and at later stages anisotropy develops.

Case (v): $n = 1$. We have here $\bar{\theta} \to$ a finite positive quantity and $|\sigma| \to 0$ at the singularity, where $\phi \to \infty$. Near the singularity $\dot{\phi} < 0$ and $|\dot{\phi}| \sim \phi$. $\dot{\psi}$, $\dot{\theta}$ and $\dot{\gamma}$ all behave like finite time-independent quantities. The expansion $\bar{\theta}$ increases in the course of time and both $\bar{\theta}$ and the anisotropy $|\sigma|$ become extremely large asymptotically when ϕ becomes vanishingly small.

Case (vi): $0<n<1$. We have here the expansion $\bar{\theta}$ zero at the singularity. There is no anisotropy ($|\sigma|=0$) at this initial stage. $\dot{\phi}$ is negative and $|\dot{\phi}| \sim \phi^{(n+1)/2}$, so that $\dot{\psi}$, $\dot{\theta}$ and $\dot{\gamma}$ all have zero magnitudes when $\phi \to \infty$. Asymptotically as $\phi \to 0$, $\bar{\theta} \to \infty$, $|\sigma| \to \infty$. Here also the dynamics of the model exhibits acceleration instead of deceleration.

For cases $n<2$ all the linear dimensions vanish near the singularity and thus the models start from point-like singularities. For cases $n>2$ or $n<0$ there is a volume collapse represented by $\bar{\theta} \to -\infty$, but all of $\dot{\psi}$, $\dot{\theta}$ and $\dot{\gamma}$ may not be of the same sign depending on constants D_1, D_2 and D_3 as can be seen from simple analysis of equations (2.19), (2.20), (3.3) and (4.2) near the singularity.

The singularities in this case may not necessarily be of point type. The occurrence of a line singularity or a disc-type singularity depends on the relative signs and magnitudes of D_1, D_2 and D_3. Further there are two distinct situations. In some cases the initial explosion is highly anisotropic and the anisotropy is removed, in the course of time, at late stages of evolution. In some other cases the model starts with an isotropic motion and anisotropy grows later.

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References

Bergman P G 1968 *Int. J. Theor. Phys.* **1** 25
Bishop N T 1976 *Mon. Not. R. Astron. Soc.* **176** 241
Brans C and Dicke R H 1961 *Phys. Rev.* **124** 925
Canuto V, Adams P G, Hsieh S H and Tsiang E 1977 *Phys. Rev.* D **16** 1643
Dirac P A M 1973 *Proc. R. Soc.* A **333** 403
Hoyle F and Narlikar J V 1971 *Nature* **233** 41
Gradshteyn I S and Ryzhik I W 1965 *Table of Integrals, Series and Products* (New York: Academic) p 71
Nordvedt K 1970 *Astrophys. J.* **161** 1059
Raychaudhuri A K 1955 *Phys. Rev.* **98** 1123
Ruban V A and Finkelstein A M *Gen. Rel. Grav.* **6** 601
Ryan M P and Shepley L C 1975 *Homogeneous Relativistic Cosmologies*, (New Jersey: Princeton University Press)
Van Flandern T C 1975 *Mon. Not. R. Astron. Soc.* **170** 333
Wagoner R V 1970 *Phys. Rev.* D **1** 3209
Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley) pp 245, 623